International Journal of Theoretical Physics, Vol. 42, No. 9, September 2003 (© 2003)

Classical Solutions From Quantum Regime for Barotropic FRW Model

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Received April 17, 2003

The quantization of gravity coupled to barotropic perfect fluid matter field with a cosmological constant is carried out. The wave function can be determined for any curvature index κ in the FRW minisuperspace model. The meaning of the existence of the classical solution is discussed in the WKB semiclassical approximation.

KEY WORDS: Classical and quantum exact solutions.

1. INTRODUCTION

It is a general belief that an initial singularity can be removed through the employment of a quantum theory of gravity. However, there is no consistent theory of gravity until now, and in this sense the problem of the initial singularity remains of actuality.

It is well known that is possible to construct a quantum model for the universe as a whole, through the Wheeler–DeWitt (WDW) equation, based in the ADM decomposition of the gravity sector, which leads to a Hamiltonian approach of general relativity, from which a canonical quantization procedure can be applied (Halliwel, 1991). Moreover, in the Hamiltonian formalism, the notion of time is lost (Isham, 1992), but there are some proposals by which this notion of time can be recovered (Schutz, 1970, 1971) by coupling of the gravity sector to a perfect fluid. In this scheme, called Schutz's formalism, a quantization procedure is possible, and the canonical momentum associated with the perfect fluid appear linearly in the WDW equation, permiting to rewrite this equation in the form a Schrödinger equation with a time coordinate associated with the matter field.

Recently, a convincing quantum gravity origin for inflation has been sought. According to Bojowald (2002) a nonperturbative approach would produce the most reliable answer as to whether or not inflation can be derived from quantum

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0020-7748/03/0900-2087/0

gravity. At the less ambitious level of minisuperspace quantum cosmology the generality of inflation as well as the quantum creation of the universe should be treated by analyzing the WDW equation and its physical solutions, the so-called wave functions of the universe (Gibbons and Grishchuk, 1989).

In this way, the behavior of the scale factor may be determined in two different forms from the quantum regime: when the wave function is time-dependent, we can calculate the expectation value of the scale factor, in the spirit of the many worlds interpretation of quantum mechanics (Tipler, 1986)

$$\langle A \rangle_t = \frac{\int_0^\infty \mathcal{W}(A)\Psi^*(A,t)A(t)\Psi(A,t)\,dA}{\int_0^\infty \mathcal{W}(A)\Psi^*(A,t)\Psi(A,t)\,dA}, \tag{1}$$

where $\mathcal{W}$(A) is a weight function that normalizes the expectation value. The other way is to apply the WKB semiclassical approximation, in which case the system follows a real trajectory given by the equation

$$\prod_q = \frac{\partial \Phi}{\partial q}, \tag{2}$$

where the index q designates one of the degrees of freedom of the system, and Φ is the phase of the wave function when written as

$$\Psi = W e^{i\Phi}, \tag{3}$$

the functions W and Φ are real functions. We follow this last procedure to obtain the classical behavior for the scale factor, using the FRW model with a barotropic perfect fluid and cosmological constant.

The work is organized as follows. In next section we describe the quantum model with a solution for any κ case in the minisuperspace, considering a barotropic perfect fluid as matter field including the cosmological constant. In Section 3, we present the classical evolution for the scale factor derived following the semiclassical WKB procedure. Section 4 is devoted to conclusions.

2. QUANTUM MODEL

The total Lagrangian for a barotropic perfect fluid coupled to gravity using the FRW geometry with the classical cosmological constant term is given by

$$\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{geom}} + \mathcal{L}_{\text{matter}} \tag{4}$$

namely

$$\mathcal{L}_{\text{geom}} = \sqrt{-(4)g}[R - 2\Lambda] = -\frac{6A^2}{N}\frac{d^2A}{dt^2} - \frac{6A}{N}\left(\frac{dA}{dt}\right)^2 + \frac{6A^2}{N^2}\frac{dA}{dt}\frac{dN}{dt} - 6\kappa NA - 2N\Lambda A^3$$

$$= \frac{d}{dt}\left(\frac{-6A^2\dot{A}}{N}\right) + \frac{6A}{N}\left(\frac{dA}{dt}\right)^2 - 6\kappa NA - 2N\Lambda A^3, \tag{5}$$

and the Lagrangian matter density (Pazos, 2000; Ryan, 1972)

$$\mathcal{L}_{\text{matter}} = \sqrt{-(4)g}\big[16\pi GN\rho\{(\gamma+1)(1+g^{km}U_kU_m)^{\frac{1}{2}} - \gamma(1+g^{km}U_kU_m)^{-\frac{1}{2}}\} - 16\pi G\rho(\gamma+1)U_mN^m\big]. \tag{6}$$

These results are obtained by employing the ADM form of the FRW metric

$$ds^2 = -N^2\,dt^2 + A^2\left[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\theta^2)\right], \tag{7}$$

where N is the lapse function, A is the scale factor of the model, and κ is the curvature index of the universe ($\kappa = 0, +1, -1$ plane, close and open, respectively).

We also make use of the usual perfect fluid energy–momentum tensor

$$T_{\mu\nu} = pg_{\mu\nu} + (p+\rho)U_\mu U_\nu, \tag{8}$$

where p, ρ, U_μ are the pressure, energy density, and the four-velocity of the cosmological fluid, respectively; and using the barotropic relationship $p = \gamma\rho$, $\gamma =$ constant, we have the solution for the energy density as a function of the scale factor of the FRW universe, in the usual way

$$\rho = \frac{M_\gamma}{A^{3(\gamma+1)}}, \tag{9}$$

where M_γ is an integration constant. In $\mathcal{L}_{\text{matter}}$ we also choose the comoving fluid (three-velocity $U_k = 0$), and the gauge $N^k = 0$, obtaining

$$\mathcal{L}_{\text{matter}} = 16\pi GNM_\gamma A^{-3\gamma}. \tag{10}$$

Thus, the total Lagrangian density has the following form:

$$\mathcal{L}_{\text{tot}} = \frac{d}{dt}\left(\frac{-6A^2\dot{A}}{N}\right) + \frac{6A}{N}\left(\frac{dA}{dt}\right)^2 - 6\kappa NA - 2N\Lambda A^3 + 16\pi GNM_\gamma A^{-3\gamma}. \tag{11}$$

Following the well-known procedure to get the canonical Hamiltonian function, we define the canonical momentum conjugate to the generalized coordinate A (scale factor) as

$$\prod\nolimits_A \equiv \frac{\partial L}{\partial \dot{A}}, \tag{12}$$

$$L = \prod\nolimits_A \dot{A} - N\mathcal{H} = \prod\nolimits_A \dot{A} - N\left[\frac{\prod_A^2}{24A} + 6\kappa A + 2\Lambda A^3 - 16\pi GM_\gamma A^{-3\gamma}\right], \tag{13}$$

when

$$\mathcal{H} = \frac{\prod_A^2}{24A} + 6\kappa A + 2\Lambda A^3 - 16\pi G M_\gamma A^{-3\gamma}. \tag{14}$$

Performing the variation of (13) with respect to N, $\frac{\delta L}{\delta N} = 0$, implies the well-known result $\mathcal{H} = 0$. Imposing the quantization condition and applying this Hamiltonian to the wave function Ψ, we obtain the WDW equation in the minisuperspace

$$\hat{\mathcal{H}}|\Psi\rangle = \frac{1}{24A}\left[-\frac{d^2}{dA^2} + 144\kappa A^2 + 48\Lambda A^4 - 384\pi G M_\gamma A^{-3\gamma+1}\right]|\Psi\rangle = 0. \tag{15}$$

Notice that in principle the order ambiguity in Eq. (15) should be taken into account. This is quite a difficult problem to be treated in all its generality, since the Hamiltonian operator in (15) must be written in a very general form to take into account all possible order, but, at least in the minimal case in which (Hartle and Hawking, 1983) (there are other possibilities depending on different considerations on the operators, see the works of Christodoulakis and Zanelli, 1984a,b or Lidsey and Moniz, 2000

$$A^{-1}\frac{d^2\Psi}{dA^2} \to A^{-1+p}\frac{d}{dA}A^{-p}\frac{d\Psi}{dA} = A^{-1}\left(\frac{d^2\Psi}{dA^2} - pA^{-1}\frac{d\Psi}{dA}\right), \tag{16}$$

where the real parameter p measures the ambiguity in the factor ordering. Therefore, the WDW equation can be written as follows:

$$-A\frac{d^2\Psi}{dA^2} + p\frac{d\Psi}{dA} - V(A)\Psi = 0, \tag{17}$$

where

$$V(A) = -48\Lambda A^5 + 384\pi G M_\gamma A^{-3\gamma+2} - 144\kappa A^3. \tag{18}$$

In the following we discuss some of the quantum solutions of Eq. (17) for particular values of the γ parameter and the parameter p of factor ordering.

2.1. Inflationary Scenario

In the inflationary era, including the cosmological constant Λ, we choose $\gamma = -1$. Notice that the WDW Eq. (17) can be written as

$$A\frac{d^2\Psi}{dA^2} - p\frac{d\Psi}{dA} + 144A^3(m^2A^2 - \kappa)\Psi = 0, \tag{19}$$

where

$$m^2 = -\frac{\Lambda}{3} + \frac{8}{3}\pi G M_{-1}. \tag{20}$$

The latter relationship leads to three possible cases

1. $m^2 > 0$[2]

The differential equation for this subcase is

$$A\Psi''_{-1} - P\Psi'_{-1} + 144\kappa A^3(m^2A^2 - \kappa)\Psi_{-1} = 0 \tag{21}$$

which, after the substitutions $v = m^2A^2 - \kappa$, $\Psi_{-1} = v^{1/2}y(v)$, and $z = \frac{4}{m^2}v^{3/2}$, turns for $p = 1$ into a Bessel equation with the general solution

$$\Psi_{-1} = (m^2A^2 - \kappa)^{\frac{1}{2}}\left[a_0 J_{\frac{1}{3}}(z) + b_0 J_{-\frac{1}{3}}(z)\right], \quad \text{where} \quad z = \frac{4}{m^2}[m^2A^2 - \kappa]^{3/2} \tag{22}$$

where a_0 and b_0 are superposition constants.

2. $m^2 < 0$

The differential equation for this subcase is

$$-A\Psi''_{-1} + p\Psi'_{-1} + 144\kappa A^3(|m^2|A^2 + \kappa)\Psi_{-1} = 0. \tag{23}$$

For $p = 1$ we have a modified Bessel equation with the general solution

$$\Psi_{-1} = (|m^2|A^2 + \kappa)^{\frac{1}{2}}\left[a_1 I_{\frac{1}{3}}(z) + b_1 K_{\frac{1}{3}}(z)\right], \quad \text{where} \quad z = \frac{4}{|m^2|}[|m^2|A^2 + \kappa]^{3/2}, \tag{24}$$

where a_1 and b_1 are superposition constants.

3. $m^2 = 0$

The differential equation for this situation is

$$-A\Psi''_{-1} + p\Psi'_{-1} + 144\kappa A^3\Psi_{-1} = 0 \tag{25}$$

which for $\kappa = 1$, has as solution the modified Bessel functions of order $\nu = \frac{1+p}{4}$

$$\Psi_{-1} = A^{2\nu}[A_0 I_\nu(6A^2) + B_0 K_\nu(6A^2)], \tag{26}$$

where A_0 and B_0 are superposition constants, while for $\kappa = -1$, the solution become to be the ordinary Bessel functions

$$\Psi_{-1} = A^{2\nu}[A_0 J_\nu(6A^2) + B_1 Y_\nu(6A^2)], \tag{27}$$

where A_1 and B_1 are superposition constants.

[2] In general m^2 is not a positive constant, see the corresponding definition (20).

2.2. Inflationary-Like Scenario, $\gamma = -\frac{1}{3}$

In this particular case, the WDW equation becomes

$$A\frac{d^2\Psi}{dA^2} - p\frac{d\Psi}{dA} - 48A^3(\Lambda A^2 - g)\Psi = 0, \quad \text{with} \quad g = 8\pi G M_{-1/3} - 3\kappa \tag{28}$$

which, after the substitutions $v = \Lambda A^2 - g$, $\Psi = v^{1/2}y(v)$, and $z = \frac{4\sqrt{3}}{3\Lambda}v^{3/2}$, yields for $p = 1$ a Bessel equation with the general solution

$$\Psi(A) = [\Lambda A^2 - g]^{1/2}\left\{C_0 I_{\frac{1}{3}}\left(\frac{4\sqrt{3}}{3\Lambda}[\Lambda A^2 - g]^{3/2}\right) + C_1 K_{\frac{1}{3}}\left(\frac{4\sqrt{3}}{3\Lambda}[\Lambda A^2 - g]^{3/2}\right)\right\}. \tag{29}$$

where C_0 and C_1 are superposition constants.

2.3. Dust Case ($\gamma = 0, \kappa = 0$)

For this case the WDW equation is

$$A\frac{d^2\Psi}{dA^2} - p\frac{d\Psi}{dA} + 48A^2\big(-\Lambda A^3 + 8\pi G M_0\big)\Psi = 0, \tag{30}$$

Making the transformations $z = \frac{8}{3}\sqrt{3\Lambda}A^3$ and $\Psi = e^{-\frac{1}{2}z}u(z)$, one gets for $u(z)$ the confluent hypergeometric equation

$$z\frac{d^2u}{dz^2} + (\gamma - z)\frac{du}{dz} - mu = 0 \tag{31}$$

where $m = \frac{2-p}{6} - \frac{16\pi G M_0}{\sqrt{3\Lambda}}$ and $\gamma = \frac{2-P}{3}$. The linear independent solutions are (Gradshteyn and Ryzhik, 1980)

$$u_1(z) = {}_1F_1(m, \gamma; z) \tag{32}$$

$$u_2(z) = z^{1-\gamma}\,{}_1F_1(m - \gamma + 1, 2 - \gamma; z) \tag{33}$$

where ${}_1F_1$ is the degenerate hypergeometric function.

Thus, the exact solution for Ψ becomes

$$\Psi(A) = e^{-\frac{1}{2}z}[A_0\, u_1(z) + B_0\, u_2(z)], \tag{34}$$

where A_0 and B_0 are superposition constants, u_1 and u_2 are the functions given in (32) and (33), respectively.

2.4. Stiff Fluid Case ($\gamma = 1, \kappa = 0$)

The WDW equation

$$A\frac{d^2\Psi}{dA^2} - p\frac{d\Psi}{dA} + 48\big(-\Lambda A^5 + 8\pi G M_1 A^{-1}\big)\,\Psi = 0, \quad (35)$$

has the following exact solution

$$\Psi(A) = A^{\frac{1+p}{2}} Z_\nu \left(\frac{4\sqrt{-3\Lambda}}{3} A^3\right), \text{ with } \nu = \frac{1}{3}\sqrt{\left(\frac{1+p}{2}\right)^2 - 384\pi\, G M_1}. \quad (36)$$

The exact expression of the wave function depends on the sign of the cosmological constant and the order ν:

- $\Lambda > 0$ and ν real, the functions Z_ν become the modified Bessel functions, either I_ν or K_ν, depending on the boundary conditions.
- $\Lambda < 0$ and ν real, the functions Z_ν turn into the ordinary Bessel functions, either J_ν or Y_ν, depending on the boundary conditions.
- $\Lambda > 0$ and ν pure imaginary, the functions Z_ν become the modified Bessel functions of pure imaginary order (Dunster, 1990), either I_ν or K_ν, depending on the boundary conditions.
- $\Lambda < 0$ and ν pure imaginary, the functions Z_ν become the modified Bessel functions of pure imaginary order (Dunster, 1990), either J_ν or Y_ν, depending on the boundary conditions.

3. THE CLASSICAL BEHAVIOR FROM WKB REGIME

Interesting results can be obtained at the level of WKB method if one performs the transformation $\prod_A \to \frac{d\Phi}{dA}$. Then, (14) becomes the Einstein–Hamilton–Jacobi equation, where Φ is the superpotential function that is related to the physical potential under consideration.

Introducing this ansatz in (14) we get

$$H = \frac{1}{24A}\left[\left(\frac{d\Phi}{dA}\right)^2 + 144\kappa A^2 + 48\Lambda A^4 - 384\pi G M_\gamma A^{-3\gamma+1}\right] = 0, \quad (37)$$

thus, we obtain

$$\frac{d\Phi}{dA} = \sqrt{-48\Lambda A^4 + 384\pi G M_\gamma A^{-3\gamma+1} - 144\kappa A^2} \quad (38)$$

the superpotential Φ has following form:

$$\Phi = \pm\int \sqrt{-48\Lambda A^4 + 384\pi G M_\gamma A^{-3\gamma+1} - 144\kappa A^2}\, dA, \quad (39)$$

This integral can be solved for particular cases of the γ parameter.

Now, employing the Eqs. (2), (12), and (38) we will obtain the evolution for the scale factor. The classical equation of motion is

$$\frac{12A}{N}\frac{dA}{dt} = \prod_A = 12A\sqrt{-\frac{1}{3}\Lambda A^2 + \frac{8}{3}\pi G M_\gamma A^{-3\gamma-1} - \kappa}, \tag{40}$$

in term of the "cosmic time" τ defined by $d\tau = N(t)\,dt$, Eq. (40) is read how (when we choose the gauge $N(t) = 1$, this cosmic time is the physical time t).

$$d\tau = \frac{dA}{\sqrt{-\frac{1}{3}\Lambda A^2 + \frac{8}{3}\pi G M_\gamma A^{-3\gamma-1} - \kappa}}. \tag{41}$$

3.1. Inflationary Scenario, $\gamma = -1$

Equation (41) for the inflation regime is written as

$$\tau - \tau_0 = \int_0^A \frac{dx}{\sqrt{\left(-\frac{1}{3}\Lambda + \frac{8}{3}\pi G M_\gamma\right) x^2 - \kappa}} = \frac{1}{\sqrt{a_1}} \ln\left[A + \sqrt{A^2 - a_2^2}\right] \tag{42}$$

where $a_1 = -\frac{1}{3}\Lambda + \frac{8}{3}\pi G M_\gamma$ and $a_2^2 = \frac{\kappa}{a_1}$.

The inverse of Eq. (42) gives us the following structure for the scale factor A(τ)

$$A(\tau) = \frac{1}{2}\left[e^{\sqrt{a_1}(\tau-\tau_0)} + a_2^2\, e^{-\sqrt{a_1}(\tau-\tau_0)}\right]. \tag{43}$$

The scale factor A(τ) will have a inflationary behavior only if the parameter $a_1 \gg 1$, then the cosmological constant have the restriction value $\Lambda < 3(\frac{8}{3}\pi G M_1 - 1)$. If the parameter $0 < a_1 < 1$, the scale factor vanishes very fast.

For other scenarios and particular values in κ and $\Lambda = 0$, we have the following:

1. For the dust case and $\kappa = 0$, we obtain the well-known solution $A \propto t^{2/3}$ for $N = 1$.
2. For $\gamma = 1$ and $\kappa = 0$, the scale factor have the traditional solution $A \propto t^{1/3}$ for $N = 1$.
3. For radiation case and $\kappa = -1$, the behavior is

$$A(\tau) = \sqrt{4(\tau - \tau_0)^2 - \frac{8}{3}\pi G M_{\frac{1}{3}}}. \tag{44}$$

For the case of $\kappa = 0$, $A \propto t^{1/2}$ for $N = 1$.

On the other hand, the master Eq. (41) can be solved for any γ if the cosmological constant vanish, defining d$\tau = A^{3\gamma+2} dT$, having the following general solution in the time T

$$A(T) = \left(\frac{3}{8\pi GM_\gamma}\right)^{\frac{1}{3\gamma+1}} \left[\left\{\frac{4}{3}\pi GM_\gamma(3\gamma+1)(T-T_0)\right\}^2 + \kappa\right]^{\frac{1}{3\gamma+1}} \tag{45}$$

The classical solutions obtained by this method perhaps coincide with these obtained solving the Einstein field equation, but is not possible to know if the quantum universe remains quantum forever for some case, because of the fact that the WDW equation is not time-dependent.

4. CONCLUSIONS

We discussed quantum cosmology from the point of view of simple models of minisuperspace. We found that the wave functions of the WDW equations in question are mostly Bessel functions. For particular values of the parameter p that measures the ambiguity of the factor ordering problem we obtained analytical results for a some scenarios of the universe possessing a cosmological constant. Only in the case of stiff fluid, the cosmological constant can be positive or negative. In the other cases, it can be only positive for physical reasons related to the scale factor or the potential under consideration. It is well known that the WDW cosmological equation is not an evolution equation and therefore the associated quantum states do not evolve in time. A possible way out of this difficulty could be to connect some parameters of the “quantum” WDW solutions with classical Einstein ones by phenomenological restrictions imposed on the superpotential function, as we did in this work. Using this method, we find the classical behavior for the scale factor, analogous to that found by solving the Einstein equations.

ACKNOWLEDGMENT

We thank H.C. Rosu for critical reading of the paper. This work was partially supported by PROMEP and GTO. University projects.

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